Surplus Structure in the Temporal Parameter: Consequences of the Mass-Shell Constraint and the N_{ref} Substitution

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Abstract

The temporal parameter $t \in \mathbb{R}$ carries three properties that no experiment has confirmed as features of physical reality: negative extension, loop-admitting topology, and reversal symmetry. We identify these as surplus structure in the sense of Weatherall [8], applying Gisin's observation on $\mathbb{R}$-modeling [6] to the temporal parameter specifically. Its operational content is N_{ref}: the accumulated state-transition count of a reference system, non-negative and monotonically non-decreasing. The caesium-133 hyperfine oscillator [10] is one realization; the framework is characterized through operational properties (formalized as axioms N1–N4 in Part II [38]) and is independent of the choice of clock.

We establish two theorems. *Theorem 1*: restricting the Wheeler–DeWitt scalar-field clock to its operationally grounded domain $\phi \in [0, \infty)$ halves the minisuperspace solution space, excluding independently contracting universes. *Theorem 2*: within the Page–Wootters framework, the process-accumulation arrow (a spectral property of the clock operator) and the thermodynamic arrow (a boundary condition on the constraint surface) are formally independent.

The Bondi k-calculus then derives the full Lorentz transformation — including the relativity of simultaneity — from three operational inputs: transition-count ratios between inertial observers, the relativity principle, and a finite signal speed c. No prior notion of time, metric, or spacetime is assumed; $1/\gamma$ is a theorem rather than an input. Closed timelike curves, parameter reversal in T-symmetry, and the block universe as the default interpretation of the formalism are shown to depend on the surplus structure of $\mathbb{R}$ and to lack independent empirical support.

1. Introduction

The temporal parameter $t \in \mathbb{R}$ has three properties its physical referent does not: negative extension, loop-admitting topology, and reversal symmetry. No measurement has ever returned a negative accumulated process count, a closed temporal loop, or a reversal of the temporal parameter. This is a domain mismatch between the mathematical surrogate and its operational ground.

This paper establishes two formal results. Theorem 1 proves that restricting the Wheeler–DeWitt scalar-field clock to its operationally grounded domain halves the minisuperspace solution space, excluding independently contracting universes. Theorem 2 proves that the process-accumulation arrow and the thermodynamic arrow are formally independent within the Page–Wootters framework. Section 8 closes the simultaneity sector: the full Lorentz transformation, including the $-vx/c^2$ term, is derived via Bondi's k-calculus from transition-count ratios between inertial observers, the relativity principle, and a finite signal speed c as the only empirical input — with no prior notion of time, metric, or spacetime assumed. Both theorems and the Bondi closure follow from the same source: the substitution of N_{ref} for t as the operational content of the temporal parameter.

Two conceptual frameworks converge in this paper. Weatherall [8] provides the formal criterion for surplus structure: mathematical features of a representation that have no physical counterpart, identified via categorical comparison of mathematical and physical theories. Gisin [6] provides the specific observation that motivates the application: the real numbers carry structure — uncountable precision, negative extension, completed infinities — that no physical measurement can access, and the choice of $\mathbb{R}$ as the mathematical language of physics is not physically innocent. Gisin's concern is primarily with determinism: if the state of the universe is specified by real numbers to infinite precision, classical determinism follows as a mathematical consequence of the language rather than as a physical discovery about nature. The present paper takes Gisin's observation in a different direction. Rather than asking what $\mathbb{R}$ does to determinism, it asks what $\mathbb{R}$ does to the temporal parameter specifically — and finds that three properties of $\mathbb{R}$ (negative extension, loop-admitting topology, reversal symmetry) produce three specific results in the physics literature (independently contracting Wheeler–DeWitt modes, closed timelike curves, and parameter reversal as a physical operation) that have no operational counterpart. Gisin observed that mathematical languages shape physical understanding; the N_{ref} framework identifies the specific shapes that $\mathbb{R}$ imposes on the temporal parameter and traces their consequences through the formalism.

The argument is clock-independent: the caesium definition is one realization of a general structure, and the results hold for any physical clock satisfying the counting axioms (Section 4). Consequences for closed timelike curves, T-symmetry, and the block universe follow from established mathematics (Choquet-Bruhat and Geroch [13, 14]; Borchers, Guido-Longo, and Mund [17, 18, 19]) without new physical assumptions. The eliminativist reading of the temporal parameter is examined philosophically in [31]; the present argument arrives at the same conclusion through mathematical results rather than philosophical analysis.

The present work connects to a broad tradition of questioning the status of the temporal parameter. Barbour's timeless mechanics [33] argues that time does not exist as a fundamental quantity; the N_{ref} framework differs in retaining time as accumulated process count rather than eliminating it. Peres [7] defines a quantum clock operationally as a system passing through a succession of distinguishable states; the N_{ref} axioms (formalized in Part II) are a direct generalization of this definition. Isham [36] and Kuchař [37] classify the many facets of the "problem of time" in quantum gravity; the Page–Wootters framework used here is one resolution of what Isham calls the "frozen formalism" problem, and the N_{ref} axioms formalize what Kuchař identifies as the conditions for successful deparametrisation.

This is the first of three papers. The physical argument — the surplus structure identification, the two headline theorems, the clock-independence analysis, and the Bondi closure — is established here. Part II [38] provides the axiomatic and categorical foundations: the N_{ref} axioms (N1–N4), the categorical surplus identification via the forgetful functor between mathematical and physical categories, the computational gauge characterization (CG1–CG3), and the derivation that the full $(1, 3)$ Minkowski signature is forced by counting, vacuum symmetry, and spatial isotropy. Part III [39] extends the framework to quantum clocks: the Bondi k-factor becomes a quantum operator, revealing a two-sector decomposition of the Lorentz transformation with structurally different quantum-correction profiles. The sector decomposition provides a structural explanation for the angular separation of quantum corrections computed independently by Grochowski et al. [30]; the framework does not predict these corrections but organizes them under a single structural cause.

2. The Resource Constraint and Operational Ground

2.1. The mass-shell relation as a conserved-total constraint

The mass-shell relation

$$E^2 = p^2c^2 + m^2c^4 \tag{1}$$

is the formal anchor of this paper. For an isolated composite system, total energy E is conserved. Equation (1) is a conserved-total resource constraint: spatial momentum pc and rest-mass energy mc^2 compete for shares of a fixed total. An increase in spatial momentum necessitates a decrease in the internal dynamics component, specifically the evolution of the system's internal degrees of freedom.

The four-velocity formulation makes the constraint explicit. The norm condition $g_{\mu\nu}U^\mu U^\nu = -c^2$ fixes every system's total four-velocity magnitude. At rest, the full norm is allocated to the temporal component. As spatial velocity increases, the temporal component decreases:

$$d\tau = dt\,\sqrt{1 - v^2/c^2} \tag{2}$$

The factor $1/\gamma = \sqrt{1 - v^2/c^2}$ is not merely kinematic shorthand. It is the fraction of the four-velocity norm remaining available for internal dynamics after spatial motion has claimed its share. At $v = 0$: full internal dynamics. At $v = c$: none.

Pikovski et al. (2015) gave this constraint a sharp Hamiltonian form in the post-Newtonian gravitational setting [1]; Smith and Ahmadi (2020) derived the exact relativistic form directly from the mass-shell constraint [2]. For a relativistic composite particle with internal Hamiltonian $\hat{H}_0$, the total Hamiltonian expanded in v/c is

$$H \approx \gamma mc^2 + \hat{H}_0/\gamma. \tag{3}$$

The factor $1/\gamma$ multiplies the internal Hamiltonian directly: the suppression of internal state-transition rates by $1/\gamma$ is the direct physical content of equation (3).

A precision on validity: the expansion (3) holds when $\hat{H}_0 \ll mc^2$, satisfied for essentially all physical systems. The $1/\gamma$ suppression of internal dynamics is a general feature of the relativistic coupling that persists at all orders. It is not an artifact of the low-energy expansion.

Equation (3) accounts for every effect attributed to "time slowing down" — muon lifetimes, atomic transition frequencies, and (via the equivalence principle) gravitational redshift and satellite-ground clock comparisons — through the $\hat{H}_0/\gamma$ suppression alone. Nothing is left over that requires an independent temporal entity whose rate varies. This is the Michelson–Morley result for the temporal parameter: just as the luminiferous ether did no independently detectable work beyond the field equations, "time slows down" does no independently detectable work beyond $\hat{H}_0/\gamma$. The removal simplifies the ontology without changing any prediction.

2.2. The budget equation as the primary object

The content of the previous subsection can be stated as a single identity. Rearranging the four-velocity norm $u^\mu u_\mu = c^2$ into the rest frame of the parametrizing clock gives

$$\left(\frac{d\tau}{dt}\right)^2 + \frac{v^2}{c^2} = 1. \tag{4}$$

We call (4) the *budget equation*. Its content is a conservation law: the total motion budget is unity, and spatial velocity and internal tick rate compete for shares of it. At $v = 0$ the entire budget is available for internal dynamics; at $v = c$ none is. Every intermediate velocity interpolates continuously between these extremes.

In Part II [38], the budget equation is not postulated but derived (the signature theorem). The counting axioms and vacuum symmetry force the invariant bilinear form to have signature $(1,1)$, so $Q(t,x) = t^2 - x^2$. The unique one-parameter subgroup of $O(1,1)$ yields the boost $\Lambda(\varphi)$, and the identity $\cosh^2\varphi - \sinh^2\varphi = 1$ is precisely (4). Here we state it and develop its consequences.

A critical framing point: the budget equation is the primary object. The Pikovski Hamiltonian $\hat{H}_0/\gamma$ is one projection of it—the Hamiltonian formulation valid for composite systems within the expansion regime $\langle \hat{H}_0 \rangle \ll mc^2$. That regime covers essentially all laboratory systems, which is why equation (3) is so broadly successful. But for a system where the expansion fails—a light clock whose photon energy is comparable to its mirror rest mass, for instance—the Hamiltonian form is approximate while the budget equation still gives $d\tau/dt = 1/\gamma$ exactly. The budget equation is exact; the Pikovski Hamiltonian is a consequence of it, not the other way around.

A structural observation clarifies why the budget equation exists at all. In Newtonian mechanics the temporal parameter is not a physical degree of freedom: it enters no Lagrangian, carries no energy, and is unaffected by the state of the system it parametrizes. A rate defined against such a parameter — $v = dx/dt$ — involves a denominator that is dynamically inert. No constraint links the denominator to the numerator. In the N_{ref} framework the temporal reference is a physical system: a composite object on the same mass-shell as the system under study. When both numerator and denominator of a rate are physical systems subject to the same constraint, the budget equation (4) is the joint condition they must satisfy. The mass-shell relation $E^2 = p^2c^2 + m^2c^4$ is not an additional postulate; it is the algebraic form of that joint condition for a system with rest mass m.

2.3. The Page–Wootters framework and the Smith–Ahmadi equivalence

The resource constraint has a quantum-mechanical grounding that is by now established.

Page and Wootters (1983) showed that a globally static quantum state $\hat{H}|\Psi\rangle = 0$ can contain a subsystem that appears to evolve from the perspective of an internal observer: time emerges as a relational phenomenon between entangled subsystems, not from an external background parameter [3]. Moreva et al. (2014) provided an experimental illustration of this relational picture using entangled photons [4].

Smith and Ahmadi (2020) proved the critical link [2]: the Page–Wootters constraint $\hat{H}|\Psi\rangle = 0$, taken in its relativistic form, is simultaneously a Wheeler–DeWitt constraint and, when deparametrized by solving for one variable to serve as the evolution parameter, the Pikovski Hamiltonian (3). PaW is the timeless constraint picture; Pikovski is the deparametrized dynamics. They are the same physics at different levels of description. Höhn, Smith and Lock (2021) extended this to a three-way equivalence with the relational Dirac observables of Rovelli's relational quantum mechanics [9, 5].

The derivational chain is therefore closed:

$$\underbrace{\hat{H}|\Psi\rangle = 0}_{\text{PaW constraint}} \longrightarrow \underbrace{E^2 = p^2c^2 + m^2c^4}_{\text{mass-shell}} \longrightarrow \longrightarrow \underbrace{\hat{H}_0/\gamma}_{\text{Pikovski}} \longrightarrow \underbrace{d\tau/dt = 1/\gamma}_{\text{time dilation}} \tag{5}$$

These results are cited as established ground. None of the steps in (4) are re-derived here. What is new in the present paper is the observation, developed in Section 3, that the operational content of the temporal parameter throughout this chain is a quantity that is non-negative and monotonically non-decreasing, and that the standard representation $t \in \mathbb{R}$ carries three additional properties that no step in (4) requires or confirms.

2.4. The SI definition and the domain mismatch

The Système International defines the second as exactly 9,192,631,770 hyperfine transitions of the ground state of caesium-133 [10]. This is not a measurement of an independently existing quantity called "time"; it is the identification of time with a physical process count. The operational content of "one second" is "this many completed transitions."

Three properties follow immediately from this definition:

Non-negativity. An accumulated transition count cannot be negative. No physical process has ever produced a negative count of completed events. $N_{\text{ref}} \geq 0$ is not a theoretical constraint. It is what "accumulated count" means.

Monotone non-decrease. Completed transitions cannot be un-completed. Each successive measurement of N_{ref} returns a value greater than or equal to the previous one. $\Delta N_{\text{ref}} \geq 0$ always.

Monotonicity ($\Delta N_{\text{ref}} \geq 0$) is constitutive rather than derived from the quantum formalism. Unitary evolution is time-reversible; the definiteness of clock readings is secured by the conditioning operation within the Page–Wootters framework — projection onto clock eigenstates within the entangled global state. Whether physical clocks acquire definite readings is the general quantum measurement problem, not a gap specific to N_{ref}. The formal treatment is given in Part II (Assumption A1) [38].

Relational character. The SI definition compares one system's transitions to another's. The second is not a duration floating free of physical process: it is a count ratio between the system under study and the caesium reference. Time, operationally, is always a comparison between two physical transition rates.

Now consider the temporal parameter as standardly represented. The parameter $t \in \mathbb{R}$ admits:

- $t < 0$: negative time, corresponding to $N_{\text{ref}} < 0$. No accumulated count is negative.
- Closed loops: t identified modulo some period, corresponding to returning to a previous N_{ref}. This is impossible by monotonicity.
- Reversal: $t \to -t$, corresponding to $\Delta N_{\text{ref}} < 0$. No completed transition has ever been un-completed.

The mismatch is exact and complete. The temporal parameter $t \in \mathbb{R}$ carries three properties: negative extension, loop-admitting topology, and reversal symmetry. These are properties the operational ground of the SI definition does not possess and has never confirmed.

This is not a philosophical observation about the "nature of time." It is a domain mismatch between a mathematical surrogate and its operational referent. The same mismatch would arise if temperature were represented by $T \in \mathbb{R}$ without the restriction $T \geq 0$. The three excess properties are candidates for surplus structure in the sense of Weatherall [8]: mathematical features of the representation that have no physical counterparts. Section 3 formalizes the substitution. Section 4 establishes clock-independence. Section 7 audits the consequences.

3. The N_{ref} Substitution

3.1. Why notation is not merely notation here

Every physics paper that writes the temporal parameter as t inherits a structure: that the parameter has the properties of the real numbers. This inheritance is never argued. It descends from Newton and is transmitted unchanged through three centuries of notation. The symbol t presents as a member of $\mathbb{R}$ and inherits all of $\mathbb{R}$'s structure: continuity, negative extension, loop-admitting topology, and reversal symmetry. This inheritance comes without any experiment having confirmed these properties as features of physical reality.

The consequences are not merely formal. Physicists, trusting the symbol, explored its inherited structure and found results that live entirely in the surplus. Closed timelike curves are trajectories that loop in the t-coordinate, a property of $\mathbb{R}$'s topology, not of any measured physical process. Parameter reversal $t \to -t$ is a symmetry of the real line, not a symmetry that has ever been instantiated as a physical operation in any laboratory. The block universe, the interpretation that all values of t exist with equal ontological status follows automatically from $\mathbb{R}$'s structure, in which every real number is equally a member of the set.

The substitution $t \to N_{\text{ref}}$ makes the domain restriction explicit at the level of the symbol. N_{ref} cannot do any of this. It represents accumulated physical process. It does not go negative; no count of completed transitions is negative. It does not loop. You cannot return to a previous accumulation by continuing to accumulate. "Traveling to a previous N_{ref}" is not a physics problem awaiting solution; it contradicts what "accumulated" means. The three surplus properties of $\mathbb{R}$ are absent from N_{ref} by construction, not by additional physical postulate.

Like Dirac notation, which encodes algebraic structure in the symbol and so makes certain errors harder to commit, N_{ref} makes the surplus visible at the level of the symbol for the temporal parameter: results depending on the surplus become identifiable on inspection.

3.2. Definition and substitution rules

Definition (Relational temporal observable). Let $\mathcal{S}_{\text{ref}}$ be a designated physical reference system. N_{ref} is the accumulated state-transition count of $\mathcal{S}_{\text{ref}}$. The caesium-133 hyperfine oscillator (the SI second) is one such realization; any monotone accumulator — a decay counter, a photon cavity, a mechanical oscillator — is equally valid. The results of this paper are independent of the choice of realization. The axiomatic characterization (N1–N4) is given in Part II [38].

$$N_{\text{ref}} \in \mathbb{N}_0, \qquad \Delta N_{\text{ref}} \geq 0. \tag{6}$$

The SI second corresponds to $\Delta N_{\text{ref}} = 9{,}192{,}631{,}770$ caesium-133 transitions. The physical character of N_{ref} is discrete: each completed transition is a distinct event counted in $\mathbb{N}_0$. The equations of physics use its continuous extension to $[0, \infty)$ as a mathematical toolkit — derivatives, path integrals, and unitary groups all require a continuous parameter. This is the same relationship as angular momentum: physically quantised in units of $\hbar$, continuously parametrized in the Hamiltonian. The discrete character is the physical constraint; the continuum is the toolkit.

A single N_{ref} realization is a counter. It accumulates a non-decreasing integer sequence satisfying N1–N4. A counter can count; it cannot tell time. Time requires a second system: a rate is a ratio, and a ratio requires two terms. In the N_{ref} framework, every temporal quantity that enters the equations of physics is a ratio between two counters: $\Delta N_A/\Delta N_{\text{ref}}$, where A is the system under study and $\mathcal{S}_{\text{ref}}$ is the reference. To tell time is to coordinate — to compare one system's accumulated count against another's. The budget equation (4) is the constraint such coordination must satisfy when both counters are physical systems on the same mass shell; the structural reason the constraint exists at all is developed in §2.2 and revisited in §9.3.

The following rules define the substitution throughout the standard formalism. In each case, the equation is unchanged; the domain of physical solutions is restricted.

Rule 1 (Coordinate time). The physical interpretation of $t \in \mathbb{R}$ is $N_{\text{ref}} \in \mathbb{N}_0$. Solutions requiring $t < 0$, t looping, or t reversing correspond to $N_{\text{ref}} < 0$, $\Delta N_{\text{ref}} < 0$, or N_{ref} returning to a previous value, none of which have operational meaning. Such solutions are mathematically well-formed and physically uninstantiated.

Rule 2 (Proper time and time dilation). The proper time τ of system A is replaced by N_A, the accumulated transition count of A. Time dilation becomes the count ratio:

$$\frac{\Delta N_A}{\Delta N_{\text{ref}}} = \frac{1}{\gamma} = \sqrt{1 - v^2/c^2}. \tag{7}$$

This is the Pikovski Hamiltonian $\hat{H}_0/\gamma$ evaluated at the level of transition counts. Equation (6) is not a reinterpretation of time dilation. It is time dilation, stated in operationally transparent notation.

Rule 3 (Velocity). Velocity $v = dx/dt$ becomes the displacement-to-transition ratio:

$$v \longrightarrow \frac{\Delta x}{\Delta N_{\text{ref}}}. \tag{8}$$

The speed of light c is the maximum spatial displacement per reference transition permitted by the mass-shell constraint, the upper bound on (7), not a property of a background medium.

Rule 4 (Schrödinger evolution). The Schrödinger equation $i\hbar\,\partial_t|\psi\rangle = H|\psi\rangle$ becomes

$$i\hbar\,\frac{d|\psi\rangle}{dN_{\text{ref}}} = H|\psi\rangle. \tag{9}$$

State evolution per reference tick. The equation is unchanged; N_{ref} replaces t as the physical label on the evolution parameter.

Rule 5 (Surplus identification). For any result R in the literature: R depends on the *operational content* of the temporal parameter if and only if R can be derived with N_{ref} in place of t. If R requires $t \in \mathbb{R}$ (specifically negative values, closed loops, or parameter reversal), then R depends on the surplus structure of $\mathbb{R}$ and is unsupported until independently confirmed by measurement.

Rule 5 is the diagnostic applied in Sections 5–7. It is not a new principle; it is Weatherall's surplus structure criterion [8] instantiated for the temporal parameter.

3.3. What the substitution does not restrict

The domain restriction $N_{\text{ref}} \geq 0$ does not prevent integration, differentiation, or any other operation over the temporal parameter. The point requires emphasis because the most common objection to domain-restricted physical quantities is that the mathematical toolkit breaks down. For N_{ref} it does not.

Physical temperature provides an instructive but imperfect precedent. The restriction $T \geq 0$ is enforced by dynamics and can be violated under extreme conditions: population inversion in certain quantum systems produces experimentally realized negative absolute temperature. The N_{ref} restriction is strictly stronger: it is definitional, not dynamical. Yet the heat equation $\partial_t T = \alpha\nabla^2 T$ takes spatial derivatives of T. Partition functions integrate $e^{-\beta H}$ over all states. Thermodynamic identities differentiate T freely. Nobody objects that $\partial T/\partial x$ requires T to "really" possess the full structure of $\mathbb{R}$ including negative extension. The domain restriction $T \geq 0$ constrains the physical interpretation of solutions. Solutions with $T<0$ are mathematically well-formed and physically uninstantiated, without constraining the mathematical operations performed on T.

Solutions with support on $t<0$ or t looping are temperature-analogue solutions: mathematically present in the toolkit, physically uninstantiated in any measurement.

The distinction is precise. Negative absolute temperature is excluded by *dynamics* and can be violated when those dynamics are circumvented. Negative N_{ref} is excluded by *definition*: "accumulated count" means completed transitions, and no count of completed transitions is negative. The N_{ref} domain restriction is not a law of nature that might be violated under extreme conditions. It is what the words mean.

4. Clock-Independence of the Resource Constraint

The preceding section established that the temporal parameter carries surplus structure relative to its operational referent N_{ref}. A natural objection arises: perhaps the argument proves only that a *particular* reference oscillator — the caesium-133 hyperfine transition — dilates, leaving open the possibility that "time itself" behaves differently from this specific clock. This section closes that objection completely.

4.1. The circularity objection stated precisely

The objection can be stated with full precision. If time dilation is the suppression of internal dynamics under the mass-shell constraint, and the SI second is defined through internal oscillations of a caesium atom, then the argument appears to prove only that the SI-defined unit dilates — not that "time" dilates. The caesium atom is a quantum system with Hamiltonian $\hat{H}_0$; the Pikovski equation (3) suppresses $\hat{H}_0$ by $1/\gamma$; the transition frequency decreases. But this is just one Hamiltonian. Is the caesium choice a prerequisite of the argument, or merely a convenience?

The question is well-posed. An answer requires examining whether clocks with fundamentally different internal Hamiltonians all yield the same $1/\gamma$ suppression, and if so, why. The next three subsections do exactly this. The Lorentz structure invoked below is derived from the counting axioms in Part II (the signature theorem) [38]; the present section treats it as established.

4.2. Three alternative clocks

Clock A: Light-bounce (photon cavity). Consider a single-mode optical cavity of proper length d with massive mirrors. The internal Hamiltonian is the cavity photon mode:

$$\hat{H}_0 = \frac{\pi\hbar c}{d}\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right). \tag{10}$$

The budget equation $E^2 = p^2c^2 + m^2c^4$ constrains the total four-momentum of the composite system (mirrors plus photon field). Internal energy $\hat{H}_0$ enters the rest mass of the composite. When the system moves at velocity v, the mass-shell constraint forces

$$\hat{H}_{\mathrm{int}} = \hat{H}_0/\gamma, \tag{11}$$

and the bounce frequency satisfies $f = f_0/\gamma$ exactly. No expansion in $\hat{H}_0/mc^2$ is needed: the budget equation is primary and exact. The Pikovski expansion (3) is a secondary approximation valid when the mirror masses satisfy $\langle\hat{H}_0\rangle \ll mc^2$, which holds for any physical cavity but fails for the idealised massless-mirror limit that appears in some textbook treatments. The exact route through the mass-shell constraint requires no such approximation.

Clock B: Radioactive decay. For an unstable nucleus, $\hat{H}_0$ is the nuclear Hamiltonian governing internal excitations. The Pikovski route is primary here: the attempt frequency for barrier penetration is an eigenvalue of $\hat{H}_0$, suppressed by $1/\gamma$ under motion. The tunnelling probability per attempt is determined by the barrier shape in the nucleus's rest frame and is Lorentz-invariant. The decay rate therefore transforms as

$$\Gamma = \Gamma_0/\gamma. \tag{12}$$

This is not a theoretical prediction awaiting confirmation. Muon lifetime measurements at $\gamma \approx 15$–29 confirm equation (12) to the precision of the experiments. The internal Hamiltonian of a muon is as different from caesium's hyperfine structure as two quantum systems can be. The $1/\gamma$ factor is identical.

Clock C: Gravitational pendulum. A mechanical oscillator with $\hat{H}_0$ describing oscillation about a gravitational equilibrium. Apply the budget equation in the instantaneously comoving inertial frame; the equivalence principle bridges to the gravitational case. The oscillation frequency transforms as $f = f_0/\gamma$. Tidal corrections modify f_0 — they are corrections to $\hat{H}_0$ from spacetime curvature — but they do not modify the $1/\gamma$ suppression factor, which comes from the mass-shell constraint applied to the composite system's four-momentum. The constraint and the curvature correction factorise: one determines how the clock is built (f_0), the other how motion suppresses it ($1/\gamma$).

Three clocks. Three radically different internal Hamiltonians. One suppression factor. The pattern demands explanation.

4.3. The bridging assumption $\mathcal{A}_{\mathrm{bridge}}$

The explanation is a single assumption, which we label $\mathcal{A}_{\mathrm{bridge}}$:

Every physical realization of N_{ref} is a composite quantum system whose Hamiltonian is bounded below and admits a center-of-mass / internal factorization.

$\mathcal{A}_{\mathrm{bridge}}$ is the bridge between the mass-shell constraint (a statement about four-momenta) and the Pikovski suppression (a statement about internal Hamiltonians). Without it, the budget equation constrains total energy but says nothing about internal dynamics. With it, the factorization $H_{\mathrm{total}} = H_{\mathrm{cm}} + \hat{H}_0/\gamma + \cdots$ follows, and the $1/\gamma$ suppression of *any* bounded-below internal Hamiltonian is automatic.

Why is $\mathcal{A}_{\mathrm{bridge}}$ physically automatic? Because clocks must be composite. A structureless point particle cannot tick: it has no internal degree of freedom to serve as a periodic process. Every physical clock — optical, nuclear, mechanical, biological — is a bound system of interacting constituents. In our universe, "composite" means quantum fields with a stable vacuum, which guarantees both the bounded-below spectrum and the cm/internal factorization. A clock that violates $\mathcal{A}_{\rm bridge}$ is not a clock that escapes time dilation. It is not a clock at all.

$\mathcal{A}_{\rm bridge}$ is the name for what the quantum universe contributes to the derivation. It is not a gap in the physics — it is the physics. Labelling it explicitly converts an implicit assumption into an auditable premise, which is the function of the exercise.

4.4. One budget equation, three projections

The mass-shell constraint $E^2 = p^2c^2 + m^2c^4$ is one equation. Its physical content appears in three apparently distinct guises, depending on which quantity is projected out:

1. **Proper time.** Evaluate the budget equation for a system's accumulated transition sequence. The elapsed proper count satisfies

$$d\tau = dt/\gamma. \tag{13}$$

   This is the standard time-dilation formula, now understood as the budget equation evaluated along a trajectory.

2. **Internal-Hamiltonian suppression.** Apply the algebraic content of the budget equation to the internal Hamiltonian $\hat{H}_0$ of a composite system satisfying $\mathcal{A}_{\rm bridge}$. The result is the Pikovski form:

$$\hat{H}_{\rm int} = \hat{H}_0/\gamma. \tag{14}$$

   This is the budget equation in its Hamiltonian projection, valid in the regime $\langle \hat{H}_0 \rangle \ll mc^2$.

3. **Light-cone geometry.** Take the null limit $m \to 0$ of the budget equation. The constraint $E^2 = p^2c^2$ fixes the speed of massless particles at c in every inertial frame. The invariance of c is the budget equation in its null projection.

These three statements are not independent physical claims. They are one constraint viewed from three angles. The hierarchy is:

$$\underbrace{E^2 = p^2c^2 + m^2c^4}_{\text{budget equation}} \longrightarrow \begin{cases} d\tau = dt/\gamma & \text{(trajectory)} \\ \hat{H}_0/\gamma & \text{(Hamiltonian)} \\ c\,\text{invariant} & \text{(null limit)} \end{cases} \tag{15}$$

Any argument that establishes $1/\gamma$ suppression in one projection establishes it in all three, because the source is a single algebraic identity. The circularity objection of §4.1 asks whether the caesium clock is special. The answer is that no clock is special: every projection of the budget equation produces $1/\gamma$, regardless of which internal Hamiltonian is being suppressed.

Remark (The null limit and the absence of $N_{\rm ref}$). The null projection ($m = 0$, $E = pc$) is the case in which $N_{\rm ref}$ ceases to exist. A free massless particle has no rest frame, no internal degrees of freedom, and no bounded-below internal Hamiltonian: it does not satisfy $\mathcal{A}_{\rm bridge}$ and cannot serve as either term of a count ratio. Crucially, a free photon at $v = c$ is non-interacting: it propagates on the null mass shell without coupling to other systems. The moment a photon interacts — by absorption into a massive system or by scattering at a vertex — it is no longer a free massless particle: its energy enters a composite system's $\hat{H}_0$, or it goes off-shell with an effective mass set by the four-momentum transfer. In either case the interacting system satisfies $\mathcal{A}_{\rm bridge}$ and participates in $N_{\rm ref}$. The budget equation at $v = c$ gives $(d\tau/dt)^2 = 0$: this is not a constraint imposed on the photon but a report that a free null particle has no internal dynamics to constrain and no interaction through which to define a count ratio. $N_{\rm ref}$ requires interaction, and interaction requires departure from the null cone.

4.5. No clock can escape $1/\gamma$

Suppose, for contradiction, that a clock $\mathcal{C}$ exists whose tick rate does not transform as f_0/γ. Then $\mathcal{C}$ could be compared with a standard clock (caesium, cavity, muon — any clock satisfying $\mathcal{A}_{\rm bridge}$) to detect a discrepancy between two co-moving oscillators. This discrepancy would be a function of the system's velocity v relative to some frame, providing a measurement of absolute velocity. But a detector of absolute velocity is precisely what the Michelson-Morley null result, and every subsequent test of Lorentz invariance, has ruled out to parts in 10^{-18} [32].

The non-dilating clock is excluded not by assumption but by the budget equation itself. The mass-shell constraint defines the physical state space: four-momenta satisfying $E^2 = p^2c^2 + m^2c^4$ are physical; those violating it are not. A clock that escapes $1/\gamma$ violates the constraint. Its states do not lie in the physical Hilbert space. It does not exist as a physical system.

The three clocks of §4.2 are projections of a single constraint onto three different internal Hamiltonians, and the constraint permits no exception. Any system satisfying $\mathcal{A}_{\rm bridge}$ — that is, any composite quantum system with a bounded-below Hamiltonian — dilates by $1/\gamma$. The caesium atom is not privileged. It is convenient.

Part III extends the analysis to the k-operator regime, showing that the sector decomposition is universal across clock types and that the $1/\gamma$ suppression persists beyond the Pikovski approximation [39].

5. Theorem 1 — The WDW Domain Restriction

5.1. Setup: the scalar field clock and the $N_{\rm ref}$ connection

Quantum cosmology routinely treats matter scalar fields as internal clocks by deparametrizing the Wheeler–DeWitt equation: one degree of freedom is singled out as the evolution variable, and the remaining degrees of freedom evolve relationally with respect to it. This deparametrisation program was developed systematically by Kuchař [37]; the N_{ref} domain restriction formalizes the boundary conditions that successful deparametrisation requires. This procedure makes the scalar field clock variable subject to the same domain boundaries identified in Section 2. If the clock variable ϕ plays the role of N_{ref} in the quantum cosmological setting, and if $N_{\text{ref}} \geq 0$ by the operational argument of Section 2, then ϕ is restricted to $[0, \infty)$. The standard treatment takes $\phi \in \mathbb{R}$. The domain mismatch identified abstractly in Section 2 here becomes a concrete restriction on the solution space of a specific equation, with calculable consequences.

Consider the simplest setting: a spatially flat Friedmann universe with a free massless scalar field ϕ as internal clock and scale factor a as the remaining degree of freedom. Deparametrizing the Wheeler–DeWitt equation by treating ϕ as the evolution parameter gives

$$i\frac{\partial \Psi(a,\phi)}{\partial \phi} = |\hat{p}_a|\, \Psi(a,\phi). \tag{16}$$

The deparametrisation-of-WDW technique is canonical in the problem-of-time literature [36, 37]; the FRW plus massless-scalar model used here is studied as the simplest tractable example of relational evolution in a gravitational context [26]. It is the setting in which the consequences of the domain restriction are most directly computable.

5.2. Theorem 1

Theorem 1 (Domain restriction halves the WDW solution space). Consider equation (9).

(i) Standard domain $\phi \in \mathbb{R}$: The general solution is

$$\Psi(a,\phi) = \int dk \big[A(k)\, e^{i(ka+|k|\phi)} + B(k)\, e^{i(ka-|k|\phi)}\big], \tag{17}$$

where $A(k)$ and $B(k)$ are independent amplitude functions. The solution space has $2N$ degrees of freedom. $A(k)$ modes have positive frequency in ϕ (expanding universe); $B(k)$ modes have negative frequency in ϕ (contracting universe). The two families are uncoupled: $B(k)$ is not determined by $A(k)$.

(ii) Restricted domain $\phi \in [0, \infty)$ with Dirichlet boundary condition $\Psi(a, 0) = 0$: Imposing the boundary condition on (10) requires

$$\int dk\, [A(k) + B(k)]\, e^{ika} = 0 \quad \text{for all } a.$$

By completeness of the Fourier basis, this forces

$$B(k) = -A(k), \tag{18}$$

giving

$$\Psi(a,\phi) = \int dk\, A(k)\, e^{ika} \cdot 2i \sin(|k|\phi). \tag{19}$$

The solution space has N degrees of freedom, exactly half those of the standard domain.

Proof. Part (i) follows by direct verification that (10) solves (9) for arbitrary $A(k), B(k) \in L^2(\mathbb{R})$. Part (ii): substituting $\phi = 0$ into (10) gives $\Psi(a,0) = \int dk\, [A(k) + B(k)]\, e^{ika}$. The Dirichlet condition requires this to vanish for all a. Since $\{e^{ika}\}$ is a complete orthogonal set in $L^2(\mathbb{R})$, the integrand must vanish pointwise: $A(k) + B(k) = 0$, giving (11). Substituting into (10) gives (12). □

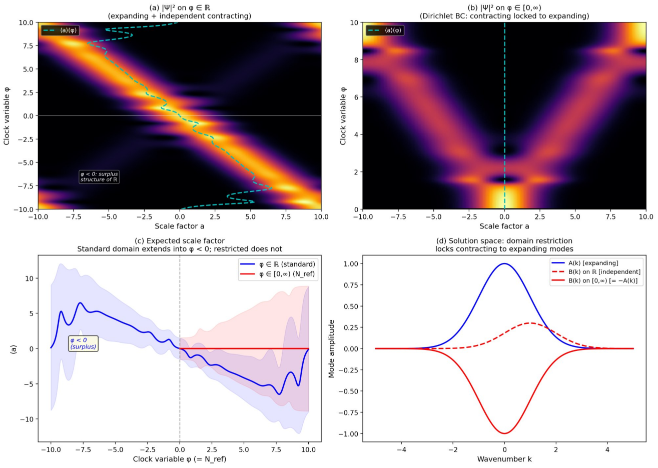

Figure 1: Wheeler–DeWitt minisuperspace calculation. **(a)** $|\Psi|^2$ on the standard domain $\phi \in \mathbb{R}$: two independent wave packets, expanding and contracting, uncorrelated at $\phi < 0$. **(b)** $|\Psi|^2$ on the restricted domain $\phi \in [0, \infty)$: components locked by the boundary condition, producing interference near $\phi = 0$. **(c)** expected scale factor $\langle a \rangle(\phi)$ on both domains. **(d)** solution space: on the standard domain $B(k)$ is independent of $A(k)$; on the restricted domain $B(k) = -A(k)$.

The Dirichlet condition $\Psi(a, 0) = 0$ corresponds to the physical requirement that no universe state exists at zero accumulated clock transitions — the constraint surface is empty before the first tick. Alternative boundary conditions (Neumann, Robin) would yield different solution spaces; the present choice is the one consistent with the operational interpretation of ϕ as accumulated process count.

Corollary. On the restricted domain $\phi \in [0, \infty)$, no independently contracting universe solution exists. Every solution contains both expanding and contracting components, locked together by condition (11). The degrees of freedom that constitute an independently contracting universe exist only in the extension of ϕ to negative values, that is, the surplus structure of $\mathbb{R}$ in the clock variable.

5.3. Physical interpretation

The theorem is a direct instantiation of the surplus structure argument at the quantum cosmological level. The independently contracting universe modes $B(k)$ are not forbidden by any dynamical equation. They are forbidden by the domain boundary: they require $\phi < 0$, which corresponds to negative accumulated clock transitions, which the operational definition of Section 2 identifies as physically uninstantiated.

The standard treatment, by taking $\phi \in \mathbb{R}$, implicitly grants physical status to $\phi < 0$ and thereby licenses the independent contracting modes. This is not a physical prediction. It is a consequence of inheriting the full structure of $\mathbb{R}$ for the clock variable without examining whether the physical referent warrants that inheritance. Theorem 1 shows the consequence of withdrawing that warrant: the solution space is halved and the independent contracting branch disappears.

Figure 1 shows the four panels of the minisuperspace calculation: $|\Psi|^2$ on the standard and restricted domains, the expected scale factor $\langle a\rangle(\phi)$, and the solution space on both domains. On the standard domain the two wave packets — one expanding ($A(k)$-dominated) and one contracting ($B(k)$-dominated) — are independent, with no correlation at $\phi<0$ where the surplus structure is active. On the restricted domain $\phi\in[0,\infty)$ they are locked together by condition (11), and the contracting branch is identified as $B(k)=-A(k)$: independently contracting universes have disappeared from the solution space.

The surplus structure of $\mathbb{R}$ in the clock variable is precisely what licenses the independent contracting modes. Under the N_{ref} domain restriction, the independently contracting universe is not a prediction of the theory. It is an artifact of the representation.

6. Theorem 2 — The Two-Arrows Independence

6.1. The conflation and its source

The arrow-of-time problem asks why the temporal direction is asymmetric. The standard reductionist account traces temporal asymmetry to boundary conditions on the initial state of the universe [34, 35]. The standard formulation treats the arrow as a single question, but it conflates two questions that are formally independent.

The first is why the clock index n is non-decreasing. The second is why entropy increases in the observed universe. These have different answers. The clock index is non-decreasing because that is a spectral property of the clock operator, built into N_{ref} by definition — a structural directedness we term the *process-accumulation arrow*. Entropy increases in our universe because the initial state of the universe had low entropy, a boundary condition on the constraint surface, the past hypothesis. Neither answer requires the other.

The single symbol t merges both questions. It presents as a parameter with a preferred direction, inviting the inference that the direction of entropy increase and the direction of temporal evolution are the same arrow and share the same explanation. They are not the same arrow. Theorem 2 proves this constructively.

6.2. Theorem 2

Theorem 2 (Process-accumulation and thermodynamic arrows are independent). Let $\mathcal{H}=\mathcal{H}_C\otimes\mathcal{H}_A\otimes\mathcal{H}_B$ be a tripartite Hilbert space with Page–Wootters constraint $\hat{H}|\Psi\rangle=0$. Let $\hat{N}$ be the clock observable with spectrum $\sigma(\hat{N})\subseteq\mathbb{N}_0$. Define the conditional state $|\psi_{AB}(n)\rangle=\langle n|\Psi\rangle/\|\langle n|\Psi\rangle\|$, the reduced state $\rho_A(n)=\mathrm{Tr}_B[|\psi_{AB}(n)\rangle\langle\psi_{AB}(n)|]$, and the von Neumann entropy $S(n)=-\mathrm{Tr}[\rho_A(n)\ln\rho_A(n)]$.

Then:

(a) n is non-negative and non-decreasing, a spectral property of $\hat{N}$, independent of the Hamiltonian and initial state.

(b) There exist states $|\Psi\rangle$ satisfying $\hat{H}|\Psi\rangle=0$ for which $S(n)$ is strictly decreasing on a finite interval while n increases throughout that interval.

Corollary. The process-accumulation arrow and the thermodynamic arrow are formally independent. The first is a structural consequence of the clock operator's spectrum. The second is determined by boundary conditions on the constraint surface, not by the direction of the temporal parameter.

Proof. Part (a) follows immediately from $\sigma(\hat{N})\subseteq\mathbb{N}_0$. Part (b) is constructive. Set $\mathcal{H}_A=\mathcal{H}_B=\mathbb{C}^2$ and define

$$H_{AB}=\omega(|00\rangle\langle 11|+|11\rangle\langle 00|),\quad \omega>0, \tag{20}$$

which acts as $\omega\sigma_x$ on the subspace $\{|00\rangle,|11\rangle\}$. Choose initial state $|\psi_0\rangle=(|00\rangle-i|11\rangle)/\sqrt{2}$, maximally entangled, $S(0)=\ln 2$.

The PaW conditional state is

$$|\psi_{AB}(n)\rangle=\frac{\cos\omega n-\sin\omega n}{\sqrt{2}}|00\rangle-\frac{i(\cos\omega n+\sin\omega n)}{\sqrt{2}}|11\rangle. \tag{21}$$

The reduced state of A is diagonal:

$$\rho_A(n)=\frac{1-\sin 2\omega n}{2}|0\rangle\langle 0|+\frac{1+\sin 2\omega n}{2}|1\rangle\langle 1|. \tag{22}$$

The entropy derivative is

$$\frac{dS}{dn}=-\omega\cos(2\omega n)\ln\frac{1+\sin 2\omega n}{1-\sin 2\omega n}. \tag{23}$$

For $n\in(0,\pi/4\omega)$: $\cos(2\omega n)>0$ and the logarithm is positive, so $dS/dn<0$ strictly throughout the interval. The entropy decreases monotonically from $S(0)=\ln 2$ to $S(\pi/4\omega)=0$ while n increases from 0 to $\pi/4\omega$. The process-accumulation arrow points forward. The thermodynamic arrow points backward. □

6.3. Genericity and figures

The constructive model is not fine-tuned. For finite-dimensional quantum systems, the independence of the two arrows follows from quasi-periodicity of unitary evolution. The eigenvalues of the evolution operator are phases $e^{i\lambda_j n}$, and the motion on the eigenvalue torus is quasi-periodic. If $S(n)$ is non-constant, it cannot increase monotonically: quasi-periodicity forces it to return arbitrarily close to its initial value, so by continuity it must decrease on some interval. The sole exception is constant $S(n)$, which requires the Hamiltonian and the initial reduced state to commute through the partial trace, a non-generic condition.

Numerical sampling confirms this. Over 500 random two-qubit Hamiltonians drawn from the Gaussian Unitary Ensemble with Haar-random initial states, 100% of models contain intervals of strictly decreasing entropy. For a four-qubit system under a random GUE Hamiltonian, 54.8% of individual evolution steps exhibit decreasing entropy.

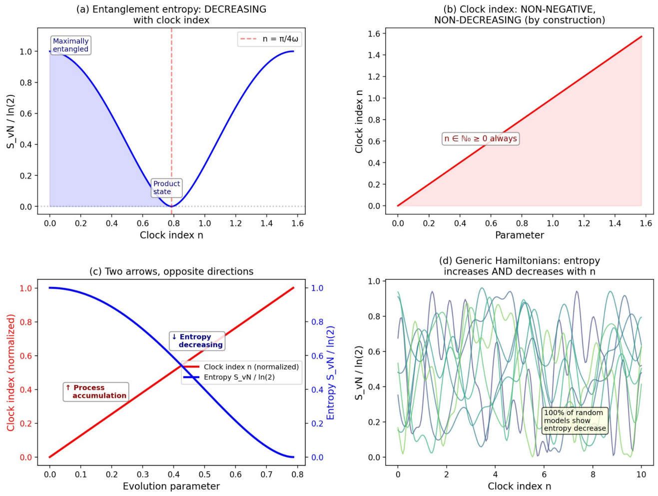

Figure 2: Two-arrow decomposition in the constructive model. **(a)** entanglement entropy $S_{\rm vN}/\ln 2$ decreasing monotonically with clock index n for $n \in (0, \pi/4\omega)$. **(b)** clock index n, non-negative and non-decreasing by construction of $\hat{N}$. **(c)** both quantities on the same axis with normalized scales: process-accumulation arrow increases, thermodynamic arrow decreases. **(d)** eight random GUE Hamiltonians, all showing entropy-decreasing intervals.

The phenomenon is a generic consequence of unitary evolution in finite-dimensional systems, not a property of specially constructed models.

Figure 2 shows the constructive model directly and Figure 3 shows the four-qubit GUE system.

6.4. Physical interpretation

The decomposition distinguishes two formally independent contributions.

The process-accumulation arrow is non-decreasing because the clock operator $\hat{N}$ has non-negative spectrum. This is not a dynamical fact. It holds for every Hamiltonian, every initial state, and every physical system. It requires no appeal to boundary conditions, statistical mechanics, or the past hypothesis. It is what $N_{\rm ref}$ means.

The thermodynamic arrow is the direction of entropy increase in the observed universe. Theorem 2 shows that this is not determined by the direction of the clock index. It is determined by the initial state of the universe, a boundary condition on the constraint surface (the past hypothesis). The two contributions are carried by different mathematical objects — a spectral property of $\hat{N}$ and a boundary condition on $|\Psi\rangle$ — and the $N_{\rm ref}$ formalism makes the separation explicit.

The definiteness of clock readings in PaW is secured by the conditioning operation, projection onto clock eigenstates within the entangled global state, not by environmental decoherence. The Pikovski $\hat{H}_0/\gamma$ coupling simultaneously generates the clock's tick rate and the decoherence that makes readings definite, without importing the thermodynamic arrow into the clock's operation.

The standard arrow-of-time problem was one problem because it was stated in terms of t. Under $N_{\rm ref}$ it becomes two problems with different answers, one structural and

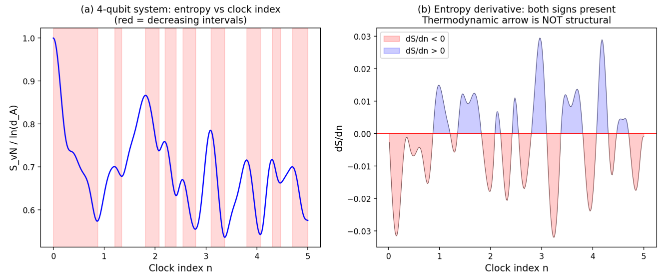

Figure 3: Four-qubit system with a Hamiltonian drawn from the Gaussian Unitary Ensemble (GUE), illustrating generic unitary dynamics. **(a)** Entropy with decreasing intervals shaded, showing the non-monotonic character of the thermodynamic arrow. **(b)** Entropy derivative dS/dn with respect to clock index n, taking both signs throughout the evolution. Together these constitute the constructive witness for Theorem 2: the thermodynamic arrow is determined by initial conditions, not by the direction of n.

one empirical, and neither answer requires the other.

7. The Surplus Structure Audit

The domain mismatch established in Section 2 and the substitution rules of Section 3 identify three $\mathbb{R}$-properties of the temporal parameter as candidates for surplus structure. Table 1 states each property, its mathematical content, the operational counter-evidence from the $N_{\rm ref}$ framework, and the physical consequence if the property is confirmed surplus. Sections 5–7 establish each consequence formally.

The three surplus properties identified in Table 1 each generate a specific result that has been taken to have physical content. This section applies Rule 5 to each: is the result derivable using $N_{\rm ref}$ in place of t, or does it depend on the surplus structure of $\mathbb{R}$?

7.1. Closed timelike curves

A closed timelike curve is a smooth causal worldline satisfying $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 0$ everywhere along its length that returns to its own starting event in the spacetime manifold. The question for the surplus structure audit is not whether such solutions can be written down — they can — but whether CTCs are *predictions* of GR in the sense that they emerge from the theory's initial value formulation, or whether their existence depends on structural features of the mathematical representation that the theory's own best-developed predictive framework excludes.

The Choquet-Bruhat–Geroch theorem. The Choquet-Bruhat theorem (1952) establishes local existence and uniqueness of solutions to the Einstein field equations from initial data on a spacelike Cauchy hypersurface [13]. Choquet-Bruhat and Geroch (1969) proved the existence of a *unique maximal globally hyperbolic development* (MGHD) for any admissible initial data set [14]. A globally hyperbolic spacetime is, by definition, one that admits a Cauchy surface — a spacelike hypersurface intersected exactly once by every inextendible causal

Table 1: Surplus structure audit of the temporal parameter.

$\mathbb{R}$-property	Mathematical content	Operational counter-evidence	Consequence (established in)
Negative extension	$t \in (-\infty, 0)$ licenses solutions with $N_{\text{ref}} < 0$	No accumulated transition count is negative; $N_{\text{ref}} \geq 0$ by definition	WDW solution space halved — independent contracting modes excluded (§5, Theorem 1)
Loop-admitting topology	$\mathbb{R}$ admits identification mod T; the t-coordinate can close on itself	No physical clock has returned to a previous transition count; $\Delta N_{\text{ref}} \geq 0$ always	CTCs fall outside GR's initial value formulation (Choquet-Bruhat–Geroch); Gödel solution requires non-constructible global matter distribution (§7.1)
Reversal symmetry	$t \to -t$ is a symmetry of $\mathbb{R}$; admits $\Delta N_{\text{ref}} < 0$	No temporal parameter reversal has been instantiated as a physical operation	T in CPT is algebraic modular conjugation J, not parameter reversal — Borchers, Guido-Longo, Mund [17, 18, 19] (§7.2)

curve. A spacetime containing a CTC admits no such surface. The MGHD is therefore CTC-free.

This result is not a coordinate statement. It operates at the level of the spacetime manifold: the unique maximal development of any admissible Cauchy data is a manifold with no closed causal curves. CTCs do not appear as outputs of GR's initial value problem.

The status of the Gödel metric. The Gödel (1949) rotating dust solution is a valid exact solution to the Einstein field equations with a physically reasonable stress-energy tensor (pressureless rotating dust with a tuned cosmological constant) [15]. Its CTC structure is genuinely topological: a worldline traversing a Gödel CTC returns to the same spacetime *event*, not merely the same coordinate label. The Gödel CTC is not a coordinate artifact.

What it is, however, is a solution outside the scope of the initial value formulation. The Gödel universe requires a specific global matter distribution that fills all of spacetime simultaneously in a rotating configuration. It cannot be prepared from a Cauchy surface: there exists no spacelike hypersurface in the Gödel spacetime that functions as initial data from which the solution can be uniquely evolved. The Gödel solution is therefore not in the domain of the Cauchy problem — it is not a prediction of GR in the sense that it cannot be derived from the theory's initial value formulation applied to any physically preparable initial state.

The Kerr interior CTC structure requires analytic extension beyond the Cauchy horizon, which is known to be unstable under generic perturbation (strong cosmic censorship); the extension is not unique and lies outside GR's predictive domain for the same reason.

The N_{ref} argument: scope and limits. A separate argument applies at the level of physical clocks. Transport a caesium clock along a Gödel CTC. At every point along the trajectory, the atom accumulates hyperfine transitions; N_{ref} increases monotonically throughout. When the spacetime coordinate returns to its starting value, the clock has accumulated additional transitions. It is not in its previous physical state.

This argument does not refute the manifold-level topology of the Gödel solution. It makes a different, weaker claim: even granting the Gödel manifold its full topological structure, a physical clock traversing the CTC has no physical realization of the "return" — its internal state has changed irreversibly, and N_{ref} does not repeat. The operational content of "returning to a previous time" — returning to a physical state indistinguishable from the departure state — is not realized by any physical clock on the Gödel trajectory.

The surplus structure claim, precisely stated. Combining the two arguments: CTCs are not derivable from GR's initial value formulation applied to physically preparable initial data (Choquet-Bruhat–Geroch). The solutions that contain CTCs — Gödel, Kerr interior — occupy exactly the sector of the solution space that the surplus structure of $\mathbb{R}$ in the temporal parameter licenses and GR's own predictive framework excludes. The claim is not that CTCs are impossible within GR as a formal system; it is that they are not *predictions* of GR and that their existence as solutions requires the loop-admitting topology of $\mathbb{R}$, which is surplus structure (Table 1, row 2). This is a weaker and more defensible claim than "CTCs are coordinate artifacts," and it is sufficient for the surplus structure argument.

7.2. T-symmetry and parameter reversal

T-symmetry has two components that must be kept separate. The first is parameter reversal: the equations of physics are symmetric under $t \to -t$. In N_{ref} notation, $N \to -N$ means negative accumulated transitions, which is operationally meaningless. This component is a property of the reversal symmetry of $\mathbb{R}$, not a property of the physical referent. The second component is dynamical invertibility: given a final state, the equations can reconstruct the initial state. The Hamiltonian generates a unitary group, and the inverse map $U(-t) = U^\dagger(t)$ exists independently of whether it is physically instantiated. Dynamical invertibility is physical and survives the N_{ref} substitution intact.

The apparent requirement for parameter reversal arises from CPT. CPT symmetry is experimentally confirmed. CP violation is observed in kaon and B-meson systems. Via the CPT theorem, T violation is inferred. This appears to require that T, and hence parameter reversal, is physically meaningful. It does not, because the CPT theorem does not require parameter reversal as an input.

The Tomita-Takesaki modular theory defines, for any von Neumann algebra of observables with a cyclic and separating vacuum vector, an anti-unitary involution J (the modular conjugation) constructed from the algebraic structure of the observable algebra. No spacetime coordinates enter the construction. The Bisognano–Wichmann theorem [16] shows that for wedge-region algebras in the vacuum representation, this algebraically defined J implements the CPT transformation. Borchers (1992) proved the algebraic CPT theorem for 2D theories [17]. Guido and Longo (1995) established it for any model satisfying modular covariance [18]. Mund (2001) extended it to massive 4D theories [19]. Swanson (2019) [20] provides a philosophical analysis of the algebraic CPT theorem's conceptual foundations. In each case, the operational content of T is carried by the anti-unitarity of J, not by the transformation $t \to -t$. The algebraic construction produces the full CPT operator; it does not decompose into independent C, P, and T factors. The combined operator is sufficient for all confirmed physical predictions; the separate $t \to -t$ reversal, while mathematically available in the real-number representation, does not appear in any derivation of observable consequences.

What T-violation experiments measure is consistent with this account. The CPLEAR (CERN, 1998) and BaBar (SLAC, 2012) experiments compared forward and reverse transition rates with appropriate quantum number conjugation. No temporal parameter was reversed in either experiment. What was measured was an asymmetry under the anti-unitary operator acting on states, a property of the S-matrix's algebraic structure.

Parameter reversal ($t \to -t$) is therefore surplus in the proof of CPT, not in the physics of CPT. The algebraic reformulation provides a proof that uses no parameter reversal as input and recovers the same theorem. Roberts (2022) [23] gives a comprehensive treatment of this distinction. The remaining open question is whether the algebraic CPT theorem extends to the full Standard Model including massless gauge fields, Haag's conjecture [22]: a technical question about the scope of the Bisognano–Wichmann property, not a conceptual challenge to the surplus structure identification.

CPT symmetry is fully intact within the N_{ref} framework. The surplus structure identification concerns the *proof strategy*, not the theorem. The algebraic CPT theorem constructs the same symmetry from modular conjugation J — a property of the observable algebra independent of any temporal coordinate. CP violation, T violation inferred via CPT, and the full experimental record are unchanged. What is surplus is the identification of T with $t \to -t$; the operational content of T is J, which requires no temporal parameter to define.

7.3. The block universe

The block universe is the interpretation that all values of t exist with equal ontological status: past, present, and future events are equally real members of the four-dimensional manifold. Its apparent inevitability comes from $\mathbb{R}$, in which every real number is a member of the set with equal standing. Equal mathematical status in $\mathbb{R}$ is not equal physical status in nature.

The standard argument for the block universe is the Rietdijk-Putnam-Penrose argument [24, 25]. Relativity of simultaneity means that what is present is observer-dependent. If two observers in relative motion disagree about which events are simultaneous with a given event, and each observer's present is equally real, then all events must be equally real. The block universe follows.

The argument conflates two claims. The first claim, that there is no observer-independent present, is a consequence of Lorentz structure and is confirmed. The second claim, that all events exist equally, is a philosophical addition. Observer-dependence of the present entails relational ontological status, not equal ontological status.

Under N_{ref} this is explicit. Different observers have different accumulated counts. The events they condition on in the PaW framework differ. No observer-independent present exists. This is relativity of simultaneity without eternalism. Future counts are what the constraint structure determines will occur, not members of a completed totality that already exists. A count of events that have not yet been completed is not a count. It is a prediction.

The block universe is not refuted by this argument. A committed eternalist can maintain the position from an external perspective. What changes is its status: before N_{ref}, the block universe appeared to follow automatically from the mathematics of $t \in \mathbb{R}$. After the substitution, it requires an independent philosophical commitment that the formalism does not supply. The burden of justification reverses.

N_{ref} is not a global temporal coordinate shared across all subsystems simultaneously. It is the eigenvalue of a local quantum observable $\hat{N}$ acting on a specific reference subsystem $\mathcal{S}_{\mathrm{ref}}$. Different observers carry different clock operators with different eigenvalue sequences; no observer-independent N_{ref} exists. This is not Newtonian absolute time with new notation. The precise relation between two observers' local N_{ref} sequences is the $(k, 1/k)$ asymmetry of Section 8: the Lorentz transformation is the bookkeeping relation between two count-based ledgers, each local, neither privileged. Lorentz covariance is preserved; a global present is not posited.

7.4. The computational gauge pattern

A natural objection to Table 1: the three properties listed are not physically idle: they do essential computational work throughout the formalism. The Feynman propagator requires negative-frequency contributions. Stone's theorem requires $t \in \mathbb{R}$ for the group inverse $U(-t) = U^{\dagger}(t)$; without it, the generator need not be self-adjoint and energy need not be a well-defined observable. Wick rotation $t \to -i\tau$ extends the parameter into the complex plane and is indispensable for lattice QCD, thermal field theory, and tunneling amplitudes.

The objection is correct and does not challenge the conclusion. The pattern is the same as for gauge redundancy in electromagnetism. The vector potential A_μ carries surplus structure: gauge freedom. QED cannot be formulated without it; it is computationally essential. But every measurable quantity is gauge-invariant: the surplus enters the calculation and exits before the observable. The temporal parameter's surplus structure behaves identically. At every level of the formalism examined, the three $\mathbb{R}$-properties enter intermediate calculations and exit before the physical prediction. This is computational gauge.

Three cases establish the pattern:

Case 1: The constraint picture. The PaW constraint $\hat{H}|\Psi\rangle = 0$ and the Wheeler–DeWitt equation contain no temporal parameter. The clock spectrum determines the ordering of conditional states. The surplus of $\mathbb{R}$ never enters. Every prediction of the timeless formalism is a correlation between subsystem states, a non-negative real number between ordered configurations.

Case 2: The algebraic CPT theorem. The Jost (1957) proof [21] constructs CPT via analytic continuation within the complexified Lorentz group, an explicit and aggressive use of the surplus. The Borchers-Guido-Longo-Mund proof [17, 18, 19] constructs the identical CPT transformation from the modular conjugation J of wedge-region algebras, without parameter reversal as input. Same theorem, two proofs: one that uses the surplus, one that does not. The surplus was in the proof, not in the physics.

Case 3: The Feynman propagator. Every internal line in every Feynman diagram includes both positive and negative frequency contributions. The negative-t surplus is used at every order of perturbation theory. The observable (the cross-section, the decay rate, the scattering amplitude) is always a non-negative real number computed between states at ordered parameter values. The Schwinger proper-time representation reparametrizes the same propagator with auxiliary parameter $s \geq 0$, recovering identical physics without the coordinate-time parameter going negative [28]. The surplus entered the Feynman representation and exited before the S-matrix element.

The pattern is systematic across all cases examined. In the deparametrized picture the surplus enters through the toolkit requirement for calculus, not through the physics. In the Wick-rotated picture the KMS condition defines thermal equilibrium algebraically [22] without requiring imaginary time; the periodicity in imaginary t is a property of the KMS state, not a physical traversal of $t < 0$. In the Feynman-Stückelberg picture the positron propagating backward in t is equivalent to a distinct particle propagating forward, with every measured property predicted identically by both descriptions.

The diagnostic (Rule 5 applied). For each $\mathbb{R}$-property in Table 1, ask whether a representation-free formulation exists that produces identical predictions without that property. If it exists, the property is computational gauge. For all three rows of Table 1, the answer is yes: the constraint picture for negative extension, the algebraic CPT theorem for reversal symmetry, and GR's initial value formulation for loop-admitting topology (established in §7.1). The surplus structure claim is therefore not falsified by the computational utility of the surplus. Utility within a representation is not evidence of physical content.

Part II formalizes this as CG1–CG3 using Stone's theorem and KMS analyticity [38].

8. The Bondi Closure — Simultaneity as a Ledger Artifact

8.1. The remaining gap, now closed

The surplus structure claim established in Sections 2 through 7 covers the time dilation sector of the Lorentz group completely. The $1/\gamma$ suppression of internal dynamics is the Pikovski Hamiltonian, derived from the mass-shell constraint, confirmed by the Smith–Ahmadi equivalence, and grounded in the SI definition. The simultaneity sector remained to be derived: the $-vx/c^2$ term in

$$t' = \gamma(t - vx/c^2), \qquad x' = \gamma(x - vt) \tag{24}$$

had not been obtained within the N_{ref} framework without importing the Lorentz group as external mathematical input. The $-vx/c^2$ term is a synchronization offset between spatially separated N_{ref} counts connected at finite speed; this section provides the quantitative derivation.

The derivation is Bondi's k-calculus [27]. Bondi's primitive objects are clock readings and light signals. These are identically the operational primitives already in use: internal state-transition counts and the SI definition of distance as light-travel count. The k-calculus was already the count-based derivation of the full Lorentz transformation. What the N_{ref} framework adds is the recognition that Bondi's primitives are not a pedagogical simplification of the standard derivation. They are the physically fundamental quantities. The derivation presented here is Bondi's; the interpretation is new.

8.2. Three ingredients, the full transformation

The derivation uses three operational inputs: transition counts N_{ref} measured by inertial observers carrying physical clocks; a finite signal speed c connecting them; and the relativity principle, realized here via the Ignatowsky conditions (homogeneity of the count space and isotropy) which the radar protocol implicitly satisfies [11]. No temporal manifold, no spatial background geometry, no synchronization convention, and no Lorentz group is assumed.

The connection to the PaW framework is direct. N_{ref} here denotes the eigenvalue of the clock observable $\hat{N}$ from Section 3, with spectrum $\sigma(\hat{N}) \subseteq \mathbb{N}_0$. The counts N_P, N'_P, N_T, N'_T are outcomes of $\hat{N}$ measurements on two physical clock systems. The Bondi derivation operates on these measurement outcomes; it is a derivation about the statistics of quantum clock eigenvalues, not a separate classical framework. The continuous limit is taken for convenience; the derivation requires only that the count ratios are well-defined.

The k-factor. Observer P sends a light signal at count N_P toward observer T receding at velocity ratio β. Define the received-to-sent count ratio:

$$k = \frac{N_T(\text{received})}{N_P(\text{sent})}. \tag{25}$$

This is a pure count ratio. It encodes two effects: classical Doppler shift from the recession, and mass-shell tick rate suppression from $1/\gamma$. P sends at N_P; T reflects at kN_P; P receives the echo at $k^2 N_P$. Radar assignment gives velocity $\beta = (k^2-1)/(k^2+1)$. Solving for k:

$$k = \sqrt{\frac{1+\beta}{1-\beta}} = (1+\beta)\gamma. \tag{26}$$

The two factors are the classical Doppler factor $(1+\beta)$ and the mass-shell suppression γ, explicit in the product.

The coordinate transformation. For an arbitrary event E, observer P sends at count N_P, receives the echo at N'_P, and assigns coordinates $t_P = (N'_P + N_P)/2$, $x_P = (N'_P - N_P)/2$. Observer T does the same. The two ledgers connect through k: outgoing signals satisfy $N_T = kN_P$, returning signals satisfy $N'_T = N'_P/k$. Substituting into T's radar coordinates and expanding using $(k+1/k) = 2\gamma$ and $(k-1/k) = 2\beta\gamma$:

$$t_T = \gamma(t_P - \beta x_P), \qquad x_T = \gamma(x_P - \beta t_P). \tag{27}$$

$$\boxed{t' = \gamma(t - vx/c^2), \qquad x' = \gamma(x - vt).} \tag{28}$$

The full Lorentz transformation, including the simultaneity term, follows from the three operational inputs above. No step required $t \in \mathbb{R}$, a spatial background, a metric, or the Lorentz group as input. The derivation works with count ratios throughout; the continuous extension to $[0, \infty)$ provides the algebraic toolkit, as described in Section 3.2.

8.3. The $-vx/c^2$ term: origin and interpretation

The origin of the simultaneity term is transparent from the derivation. When P assigns a temporal coordinate to a distant event, P uses the midpoint of send and receive counts: $t_P = (N'_P + N_P)/2$. When T does the same, T's midpoint differs because k stretches outgoing counts by a factor of k and compresses returning counts by $1/k$. The asymmetry scales with spatial separation: more distant events require longer signal travel, and k acts over a longer baseline.

The $-vx/c^2$ term is the accumulated $(k, 1/k)$ asymmetry over the spatial baseline x. It is not a property of time. It is not a property of space. It is a property of how two count-based measurement ledgers, connected by light signals and ticking at mass-shell-constrained rates, necessarily disagree about the midpoint assignment for distant events.

Einstein's train [12] makes this concrete (Figure 4). Lightning strikes both ends of a moving train simultaneously

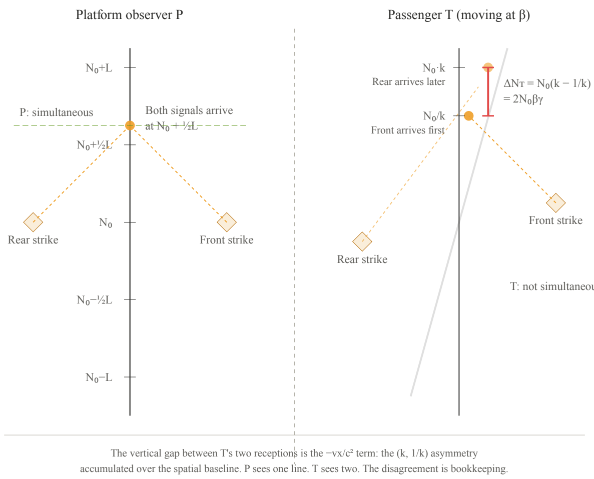

Figure 4: Einstein's train in k-calculus. The platform observer P receives both lightning signals at the same count. The train passenger T moves toward the front strike ($k < 1$, approaching) and away from the rear ($k > 1$, receding), so the two signals arrive at different T-counts. The simultaneity disagreement is arithmetically forced by the k-factor over the spatial baseline.

in the platform frame: the platform observer P receives both signals at the same count, equidistant from both strikes. The train passenger T, at the midpoint of the train, moves toward the front strike and away from the rear. The front signal arrives at a lower T-count because $k < 1$ for the approaching signal; the rear signal arrives at a higher T-count because $k > 1$ for the receding signal. The simultaneity disagreement is the difference between two midpoint assignments, a counting asymmetry between two ledgers measuring the same pair of physical events. No manifold geometry is required. No convention is chosen. The disagreement is arithmetically forced by the k-factor and the spatial baseline.

8.4. What this establishes

The surplus structure claim now extends to the full Lorentz group. Every component of (21) is a bookkeeping relation between count-based radar ledgers:

Time dilation: the diagonal term, from $1/\gamma$ tick rate suppression in the mass-shell constraint. Length contraction: the spatial consequence of the same suppression. Relativity of simultaneity: the off-diagonal term, from the $(k, 1/k)$ asymmetry accumulated over the spatial baseline.

The qualifier that the surplus structure claim was established for the time dilation sector and structurally supported for the full group is removed. The continuous manifold representation with $t \in \mathbb{R}$ is computational gauge throughout the full Lorentz group, not only in the time dilation sector.

The Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ was not assumed anywhere in the derivation. The minus sign that distinguishes the temporal component from the spatial components emerges from the $(k, 1/k)$ asymmetry — the formal proof that the non-compactness of the boost group forces indefinite signature is given in Part II (the signature theorem) [38]. It is not a primitive feature of a four-dimensional manifold. It is the algebra of how counting systems constrained by the mass-shell relation translate each other's ledgers.

8.5. Surplus structure audit of the derivation

We apply Rule 5 to each component of the Lorentz transformation explicitly, confirming that the derivation of Section 8.2 avoids each of the three surplus properties identified in Table 1.

Negative extension ($t<0$). The derivation operates entirely with count ratios N_P, N'_P, N_T, N'_T, all of which are non-negative by definition (they are accumulated state-transition counts). The k-factor is defined as $k=N_T(\text{received})/N_P(\text{sent})$, a ratio of two non-negative quantities. The radar coordinate assignments $t_P=(N'_P+N_P)/2$ and $x_P=(N'_P-N_P)/2$ can yield negative x_P (if the echo arrives before the send, which cannot happen) but never negative t_P, since $N'_P \geq N_P$ always. No step invokes a solution with $t<0$. The full Lorentz transformation (equation 28) is recovered without any count ever going negative. Negative extension is not required.

Loop-admitting topology. The derivation is acyclic by construction: P sends at N_P, T receives at $kN_P > N_P$, T reflects, P receives at $k^2N_P > kN_P$. The count index strictly increases at every step. The transformation relates two observers' coordinate assignments to the *same* set of events; it does not require any count to return to a previous value. Loop-admitting topology is not required.

Reversal symmetry ($t\to -t$). The k-calculus is asymmetric by construction: $k\neq 1/k$ for $\beta\neq 0$. The asymmetry between k (outgoing) and $1/k$ (returning) is precisely the content of the $-vx/c^2$ term: it is the accumulated $(k,1/k)$ asymmetry over the spatial baseline. The derivation does not use $t\to -t$ at any step. The sign of β enters through the convention $\beta\to-\beta$ in equation (28), which corresponds to relabeling which observer is called P and which T; it is a notational choice, not a physical reversal of the temporal parameter. Reversal symmetry is not required.

The conclusion follows directly. The full Lorentz transformation, including the $-vx/c^2$ simultaneity term, is derivable from the operational inputs of Section 8.2, without any step that requires the temporal parameter to be negative, to loop, or to reverse. The surplus structure of $\mathbb{R}$ in t is therefore computational gauge throughout the entire Lorentz group, not only in the time dilation sector previously established. This is the claim of Section 8; the audit above establishes its internal consistency.

8.6. The three roles of the temporal parameter

The Bondi closure completes the surplus structure claim for the full Lorentz group. It also makes explicit the layered structure anticipated in Section 3.2: counting is a property of one system; time is a ratio between two; coordinate time is a protocol among many. The parameter t plays three distinct roles in physics, each of which bottoms out in N_{ref}.

Local N_{ref}. In most of physics — the Schrödinger equation, Hamilton's equations, decay rates, transition frequencies, oscillation periods, statistical mechanics, fluid dynamics, Maxwell's equations in a single frame — the parameter t is the continuous extension of a local N_{ref}. No second observer is present. No coordination occurs. What the physicist calls t in these contexts is the count of transitions accumulated by the lab clock, continuously extended to permit derivatives and path integrals. The $\mathbb{R}$ structure is the mathematical toolkit (formalized as CG1 in Part II [38]); the operational content is local N_{ref}. This accounts for the large majority of appearances of t in the physics literature.

Coordination. In the multi-observer setting of this section, t plays a different role. The coordinate time that P assigns to a distant event is not P's local N_{ref} — it is P's estimate of what N_{ref} would read at that event, inferred from signal round-trips. It is a coordination artifact constructed from pairwise N_{ref} agreements via the Bondi protocol. For N observers, the construction extends by transitivity of the k-factor (Part II, the k-factor transitivity lemma [38]). The $\mathbb{R}$ domain of coordinate time is the domain of the coordination protocol under unlimited extension: negative values correspond to assignments before any physical N_{ref} began counting; reversals correspond to running the protocol backward; loops correspond to assignments that return to previous ledger entries. None of these are physical.

Surplus and toolkit. The surplus structure enters through both routes. The mathematical toolkit requires it (Stone's theorem, Feynman propagators, Wick rotation — formalized as CG1–CG3 in Part II). The coordination protocol extends into it (negative coordinate times, backward-running ledger agreements). In both cases, the extensions have no physical N_{ref} counterpart. The surplus is not an incidental feature of a particular formalism; it is the systematic structure that the $\mathbb{R}$ toolkit and the coordination protocol jointly produce when extended beyond the domain of physical counting.

The three roles of t are not three different quantities. They are one quantity — accumulated physical process count — viewed at three levels of description: local measurement, inter-observer coordination, and mathematical extension. The operational content at every level is exhausted by N_{ref}. The three-way partition is formalized in Part II (§3.10) [38]. The quantum corrections of Part III [39] operate at both the local and coordination levels.

9. Discussion

9.1. What the geometric ontology gets right

The standard geometric formulation of physics is not superseded by this analysis. It is a valid limiting description within the resource constraint framework, in the same way that Newtonian mechanics is a valid limiting description within relativistic mechanics. Every empirically confirmed prediction of the standard formulation is preserved under the N_{ref} substitution. The equations are unchanged. The domain of physical solutions is restricted. Within the confirmed domain, the geometric and resource constraint descriptions are operationally identical.

What the N_{ref} framework provides is a precise account of where the domain boundaries are. Closed timelike curves, parameter reversal as a physical operation, and the block universe as the default reading of the formalism all lie outside the empirically confirmed domain. They occupy the surplus structure of $\mathbb{R}$, the three properties listed in Table 1, none of which any experiment has ever confirmed as features of physical reality. The geometric ontology works within its domain. The N_{ref} framework identifies what lies at its edges.

9.2. Summary of the audit

Table 1 listed three surplus properties and three consequences. Each has been established by the end of Section 7. Negative extension of $\mathbb{R}$: the WDW solution space is halved under the domain restriction (Theorem 1, Section 5), and independently contracting universe solutions exist only in the surplus sector. Loop-admitting topology: CTCs fall outside GR's initial value formulation and depend on the surplus topology of $\mathbb{R}$ in the Gödel and Kerr cases (Section 7.1). Reversal symmetry: the operational content of T in CPT is carried by the modular conjugation J, not by parameter reversal (Borchers-Guido-Longo-Mund, Section 7.2), and the block universe loses its status as the default reading of the formalism (Section 7.3).

In each case examined, the surplus enters intermediate calculations and exits before every physical observable. That is the definition of computational gauge. The surplus structure of $\mathbb{R}$ in the temporal parameter is essential to the mathematical toolkit and absent from the operational content, in the same way that gauge freedom in electromagnetism is essential to QED and absent from every cross-section.

9.3. Why the constraint exists

A question remains about the status of the budget equation itself. The standard presentation takes the mass-shell relation as a Casimir invariant of the Poincaré group and derives the budget equation from it. In this paper the logical direction is reversed: the budget equation is derived from counting and vacuum symmetry (Part II, the signature theorem [38]), and the mass-shell relation is one of its projections (Section 4.4). But neither presentation explains why a constraint of this form exists in the first place.

The N_{ref} framework offers a structural answer. In Newtonian mechanics the temporal reference is dynamically inert — an abstract parameter with no mass, no energy, and no coupling to the system it parametrizes. Rates defined against such a parameter are unconstrained: a Newtonian system may move at any speed with no consequence for its internal dynamics. There is no budget equation because the denominator of every rate is free.

When the temporal reference is required to be a physical system — a composite object satisfying $\mathcal{A}_{\mathrm{bridge}}$, residing on the same constraint surface as the systems it measures — the situation changes. Two physical systems computing count ratios between each other are both subject to the mass-shell relation, and those ratios compose multiplicatively (the transitivity of count ratios, formalized as the k-factor transitivity lemma in Part II [38]). The resulting multiplicative group is non-compact, forcing an indefinite invariant form, which is the budget equation. The derivational chain is: physical denominators $\to$ constrained count ratios $\to$ multiplicative group $\to$ non-compact structure $\to$ indefinite form $\to$ budget equation $\to$ mass-shell relation. Each step is either definitional (what ratios do), empirical (no maximum boost), or a theorem of group theory (the non-compact indefinite form lemma of Part II [38]). The constraint exists because the denominator of every physical rate is itself physical.

The Newtonian limit is recovered when the denominator becomes abstract — when the reference clock is taken to be ideal, massless, and dynamically decoupled from the system. In that limit the budget equation reduces to $(d\tau/dt)^2 = 1$, the constraint becomes trivial, and spatial motion ceases to compete with internal dynamics. The Galilean group, with its unconstrained velocity addition, is the symmetry group of free denominators.

9.4. The revised hierarchy

The clock-independence analysis of Section 4 and the budget equation foregrounding of Section 2.2 establish a revised logical hierarchy for the framework's core claims. The complete input–output chain is:

Inputs: counting axioms N1–N4; relativity principle (inertial composition); spatial isotropy; spatial homogeneity; quantum mechanics with a stable vacuum ($\mathcal{A}_{\mathrm{bridge}}$ — physically automatic).

Derived (Part II, the group-property propositions and signature theorem [38]): k-factor composition $\to$ non-compact Lie group $\to$ indefinite bilinear form $\to$ budget equation $(d\tau/dt)^2 + v^2/c^2 = 1$.

Applied to any N_{ref} realization: clock must be composite (structureless particles cannot tick) $\to$ in our universe, composite = quantum fields + stable vacuum = $\mathcal{A}_{\mathrm{bridge}} \to \hat{H}_0$ identified (whatever internal dynamics the clock counts) $\to$ mass-shell constraint: $\hat{H}_{\mathrm{phys}} = \hat{H}_0/\gamma$ on Σ (exact via budget equation; Pikovski route approximate when $\hat{H}_0 \ll mc^2$).

Consequence: all clocks in our universe dilate by $1/\gamma$ (mean rate), by the same mechanism: the budget equation limits internal transition rate. The specific route (geometry, Hamiltonian, proper time) is a projection of this single constraint.

The three-way partition of Section 8.6 — local N_{ref}, coordination, surplus — locates each appearance of t in this hierarchy. Local uses are the measurement layer; coordination uses are the bookkeeping layer; surplus uses are the toolkit layer. All three bottom out in N_{ref}.

9.5. Open problems

Three problems remain precisely scoped.

The first is the quantum extension of the Bondi result. Section 8 establishes the Lorentz transformation from N_{ref} counts at the kinematic level. Part III [39] develops this extension: the k-factor becomes a quantum operator with a two-sector decomposition, and the resulting algebra produces quantum corrections with experimentally distinguishable signatures.

The second is Haag's conjecture [22]. The algebraic CPT theorem is established for 2D theories, theories satisfying modular covariance, and massive 4D theories. Whether it extends to the full Standard Model including massless gauge fields is a technical question about the scope of the Bisognano–Wichmann property whose resolution will either complete or bound the surplus structure identification for parameter reversal.

The third concerns experimental contact. The N_{ref} framework does not modify the equations of physics and therefore does not, by itself, predict deviations from standard relativistic quantum mechanics. Its contribution is structural: it identifies which features of the formalism carry physical content and which are surplus, and it organizes known quantum corrections (the sector decomposition of Part III [39]) under a single structural cause. The framework would acquire direct experimental leverage if the domain restriction of the temporal parameter produced observable consequences — as Theorem 1 demonstrates for the WDW minisuperspace, where the restricted domain halves the solution space. Whether analogous domain-restriction effects arise in laboratory quantum systems is open. Separately, the quantum corrections organized by the sector decomposition — distinguishable by angular detection geometry as shown by Grochowski et al. [30] — provide experimental tests of the structural claims of Part III, and distributed quantum clock networks such as those proposed by Covey, Pikovski and Borregaard [29] offer a route to observing quantum time dilation in curved spacetime.

10. Conclusion

The temporal parameter $t \in \mathbb{R}$ carries three properties: negative extension, loop-admitting topology, and reversal symmetry. No experiment has confirmed any of them as features of physical reality. The operational content of t is N_{ref}: the accumulated state-transition count of a reference system, non-negative and monotonically non-decreasing. The SI second is one realization; the axiomatic characterization (N1–N4) is independent of the choice of clock.

Theorem 1 establishes that restricting the Wheeler–DeWitt scalar-field clock to its operationally grounded domain $\phi \in [0, \infty)$ halves the minisuperspace solution space. Independent contracting universe solutions exist only in the surplus structure of $\mathbb{R}$.

Theorem 2 establishes that the process-accumulation arrow, a spectral property of the clock operator, and the thermodynamic arrow, determined by boundary conditions on the constraint surface, are formally independent. The single symbol t merged them. The symbol N_{ref} makes them separate.

Section 8 establishes that the full Lorentz transformation, including the $-vx/c^2$ simultaneity term, is derivable from transition counts, light signals, the mass-shell constraint, and the Ignatowsky conditions — operationally accessible inputs far weaker than full Poincaré covariance. The surplus structure claim extends to the full Lorentz group.

The parameter t plays three roles — local clock count, inter-observer coordination, and mathematical toolkit extension — each of which is exhausted by N_{ref} (Section 8.6).

Results depending on the surplus (closed timelike curves, parameter reversal as a physical operation, and the block universe as the default reading of the formalism) are not predictions of physics. They are consequences of modeling the temporal parameter with $\mathbb{R}$ and inheriting its structure without examination. They require independent empirical confirmation that does not exist.

Part II [38] provides the axiomatic and categorical foundations of this framework. Part III [39] extends the Bondi closure to quantum clocks, developing the k-operator algebra and its experimental consequences.

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Data availability. No datasets were generated or analysed during the current study. All results are derived analytically from the formalism developed in the text; the numerical illustrations in Figures 1, 2, and 3 were produced by direct computation from the definitions stated in the paper and can be reproduced from the code available from the author on reasonable request.

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